Localization Method for a Multivariate Ensemble Kalman Filter

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May 20, 2014

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Motivation

- Localization by Schur (elementwise) product of $\mathbf{P}^b$ and a localization matrix from a compactly supported correlation function $\rho(\cdot)$
- In statistics, such localization function is often called "taper"; needs to be positive definite
- For multivariate state variables, current practice is to apply the same localization function to each "block" of $\mathbf{P}^b$

Motivation

- For example (bivariate case):
  - Bivariate ensembles $\mathbf{x}^{b(k)} = (\mathbf{x}_1^{b(k)}, \mathbf{x}_2^{b(k)})$, $k = 1, \ldots, M$
  - $\mathbf{P}^b = \frac{1}{M-1}\mathbf{X}^b\mathbf{X}^{bT}$, where $\mathbf{X}^b = \mathbf{x}^{b(k)} - \bar{\mathbf{x}}^{b(k)}$
  - $\mathbf{P}^b$ can be expressed as

$$\mathbf{P}^b = \begin{pmatrix} \mathbf{P}_{11}^b & \mathbf{P}_{12}^b \\ \mathbf{P}_{21}^b & \mathbf{P}_{22}^b \end{pmatrix},$$

where $\mathbf{P}_{ij}^b = \frac{1}{M-1}\mathbf{X}_i^b\mathbf{X}_j^{bT}$ and $\mathbf{X}_i^b = \mathbf{x}_i^{b(k)} - \bar{\mathbf{x}}_i^{b(k)}$

Motivation

- Problem of rank deficiency:
  - Localization matrix $\begin{pmatrix} L & L \\ L & L \end{pmatrix}$
  - Problem is more serious when $\mathbf{P}^b_{ij}$'s are "significant"
- Mathematically, we need $\boldsymbol{\rho}(\cdot) = \{\rho_{ij}(\cdot)\}_{i,j=1,\dots,N}$: matrix-valued correlation (positive definite) function, N : number of state variables
- Even in statistics literature, not many known such "valid" $\boldsymbol{\rho}$ (parametric) functions are available yet!

One idea

- Use $\rho_{ij}(\cdot) = \beta_{ij} \cdot \rho(\cdot)$ with $|\beta_{ij}| < 1$, $|\beta_{ji}| < 1$, and $\beta_{ii} = \beta_{jj} = 1$.
  - $\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$ is positive-definite and of full rank for any β with $|\beta| < 1$.
  - For ρ, use any localization functions in Gaspari and Cohn (1999).
- This multivariate localization function is separable in the sense that

  multivariate component (in the above example, $\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$)

  and

  localization function (in the above example, ρ)

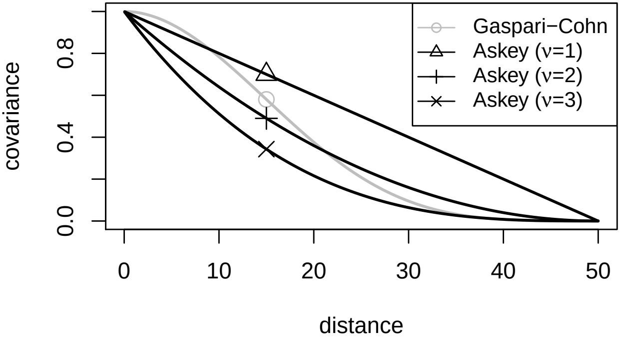
  are factored: no interaction!

Another idea

- Use one of a few multivariate compactly supported correlation functions available in statistics literature.
  - e.g. Bivariate Askey function (Porcu et al. 2012)

$$\rho_{ij}(d;\nu,c) = \beta_{ij}\left(1-\frac{d}{c}\right)_{+}^{\nu+\mu_{ij}},$$

  - $c > 0$, $\mu_{12} = \mu_{21} \geq \frac{1}{2}(\mu_{11}+\mu_{22})$, $\nu \geq [\frac{1}{2}s]+2$, and s is space dimension.
  - $|\beta_{ij}| \leq \frac{\Gamma(1+\mu_{12})}{\Gamma(1+\nu+\mu_{12})}\sqrt{\frac{\Gamma(1+\nu+\mu_{11})\Gamma(1+\nu+\mu_{22})}{\Gamma(1+\mu_{11})\Gamma(1+\mu_{22})}}$, $\beta_{ii} = \beta_{jj} = 1$
  - $|\beta_{ij}| \leq 1$ if $\mu_{11} = \mu_{22}$.


Gaspari−Cohn
Askey (ν=1)
Askey (ν=2)
Askey (ν=3)
covariance
0.0
0.4
0.8
0
10
20
30
40
50
distance

Experiment with bivariate Lorenz Model

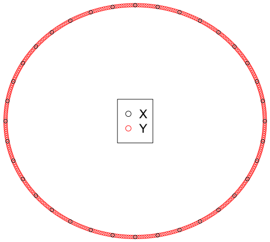
- X_k and $Y_{j,k}$ are equally spaced on a latitude circle ($j = 1, \ldots, J$ and $k = 1, \ldots, K$).
- With boundary conditions $X_{k \pm K} = X_K$, $Y_{j,k \pm K} = Y_{j,k}$, $Y_{j-J,k} = Y_{j,k-1}$, and $Y_{j+J,k} = Y_{j,k+1}$,

$$\frac{dX_k}{dt} = -X_{k-1}(X_{k-2} - X_{k+1}) - X_k - (ha/b)\sum_{j=1}^{J} Y_{j,k} + F,$$

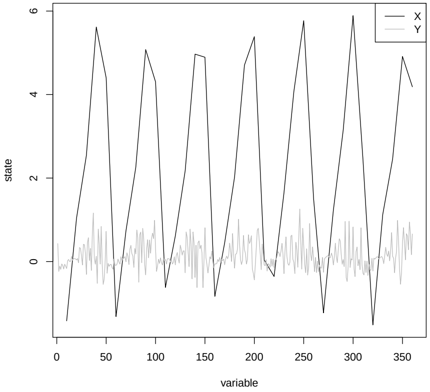
$$\frac{dY_{j,k}}{dt} = -abY_{j+1,k}(Y_{j+2,k} - Y_{j-1,k}) - aY_{j,k} + (ha/b)X_k$$

Bivariate Lorenz Model

- 36 variables of X, 360 variables of Y, $a = 10$, $b = 10$, $h = 2$



locations



longitudinal profiles

Experiment set up

- Two scenarios for observation
  1. Observe 20% of $\mathbf{X}$ and 90% of $\mathbf{Y}$ at locations where $\mathbf{X}$ is not observed.
  2. Fully observe $\mathbf{X}$ and $\mathbf{Y}$

- Four localization schemes
  - S1 No localization
  - S2 No localization and let $\mathbf{P}^b_{12} = \mathbf{P}^b_{21} = 0$.
  - S3 localize $\mathbf{P}^b_{11}$ and $\mathbf{P}^b_{22}$, but let $\mathbf{P}^b_{12} = \mathbf{P}^b_{21} = 0$.
  - S4 localize $\mathbf{P}^b_{11}$, $\mathbf{P}^b_{22}$, $\mathbf{P}^b_{12}$, $\mathbf{P}^b_{21}$

Localization (S4)

1 Gaspari-Cohn function: $\rho_{ij}(d;c) = \beta_{ij}\rho(d;c),\ i,j = 1,2$, where

$$\rho(d;c) = \begin{cases} -\frac{1}{4}(|d|/c)^5+\frac{1}{2}(d/c)^4+\frac{5}{8}(|d|/c)^3-\frac{5}{3}(d/c)^2+1, & 0\le|d|\le c; \\ \frac{1}{12}(|d|/c)^5-\frac{1}{2}(d/c)^4+\frac{5}{8}(|d|/c)^3+\frac{5}{3}(d/c)^2-5(|d|/c)+4-\frac{2}{3}c/|d|, & c\le|d|\le 2c; \\ 0, & 2c\le|d| \end{cases}$$

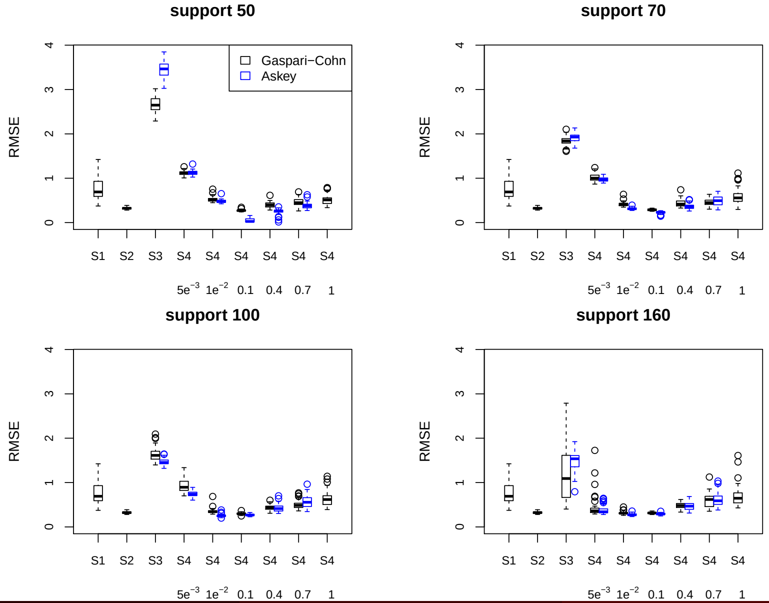
and $\beta_{11} = \beta_{22} = 1$, $0 \le \beta_{ij} \le 1$. (support=$2c$)

2 Bivariate Askey function

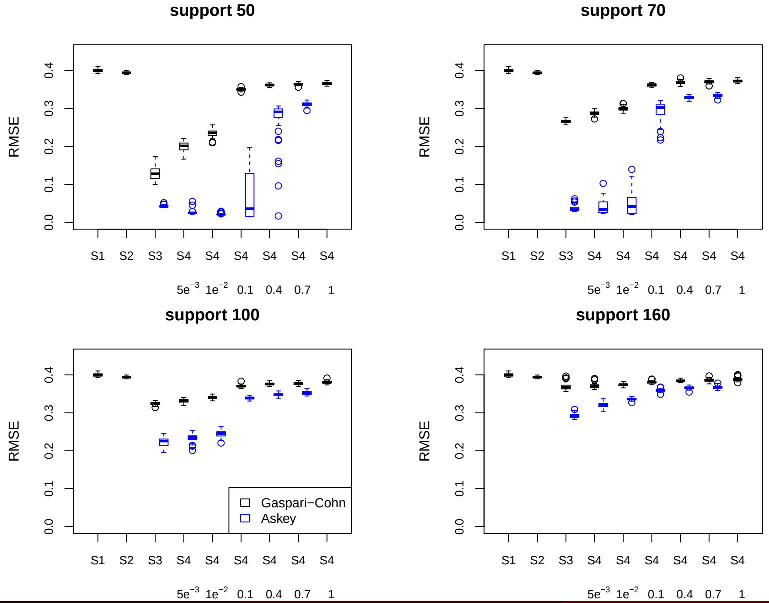
$$\rho_{ij}(d;c) = \beta_{ij}\left(1 - \frac{|d|}{c}\right)_+^{\nu+\mu_{ij}},\ i,j = 1,2$$

with $\mu_{11} = 0$, $\mu_{22} = 2$, $\mu_{ij} = 1$, $\nu = 3$, and $\beta_{11} = \beta_{22} = 1$, $0 \le \beta_{ij} \le 0.7$. (support=c)

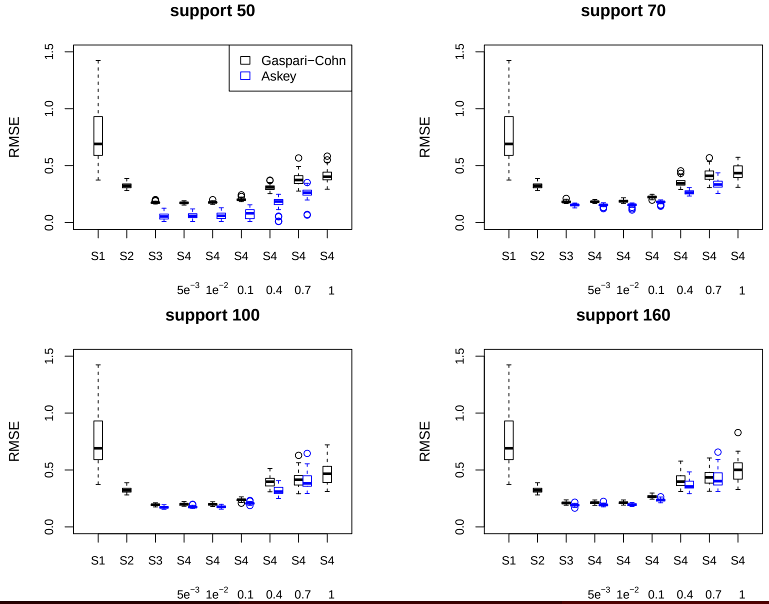
Results for **X** in scenario 1



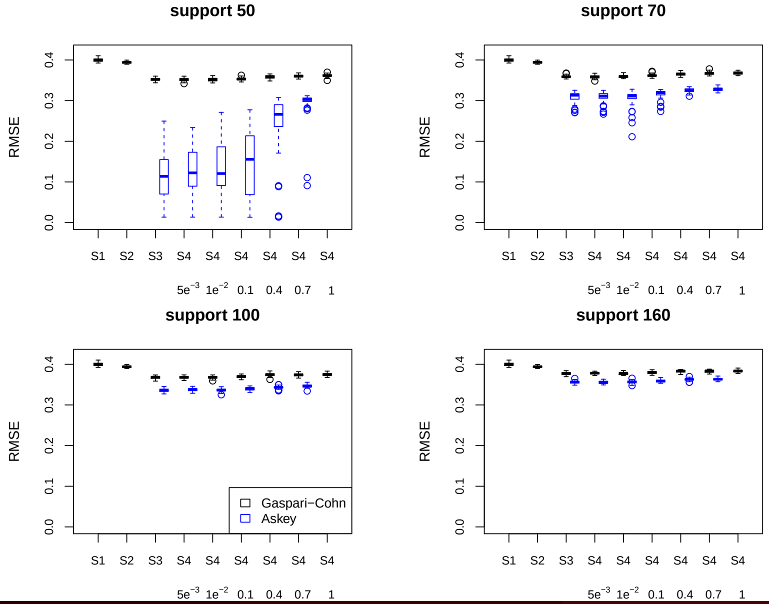
Results for **Y** in scenario 1



Results for **X** in scenario 2



Results for **Y** in scenario 2



Some issues

- Different localization length for each state variable?
- Estimation of “tuning parameters”

Thank you!

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