The Kalman-Bucy Filter in the ensemble framework

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Abstract

In this work, we show that (a) the **Kalman-Bucy Filter** (KBF) **can** indeed **be used in EnKF applications**, (b) the **stiffness found in the 'pseudo-time' integration** required in the Ensemble Kalman-Bucy filter **is overcome with a Diagonal Semi Implicit** (DSI) **scheme**, c) a **new ensemble transform formulation** is efficient for both the perturbations and for the full ensemble, (d) the **performance of the new ensemble KBFs is comparable to** the highly efficient **LETKF** (Hunt *et al.*, 2007).

Ensemble Kalman-Bucy Filters

The **Kalman-Bucy filter** (KBF, Kalman and Bucy, 1961) can be used in an **ensemble** data assimilation framework **with discrete-time observations**.

Let $\mathbf{x} \in \Re^N$ represent the **state variables**, $\mathbf{y} \in \Re^L$ the **observations** and $H \in \Re^{L \times N}$ the observational matrix operator.

The **ensemble** can be represented as $\overline{\overline{\mathbf{X}}} = [\mathbf{x}_1 | \mathbf{x}_2 | \cdots | \mathbf{x}_m] \in \Re^{N \times M}$ and the **ensemble of perturbations** as: $\mathbf{X} = [\mathbf{x}_1 - \overline{\mathbf{x}} | \mathbf{x}_2 - \overline{\mathbf{x}} | \cdots | \mathbf{x}_M - \overline{\mathbf{x}}]$, where $\overline{\mathbf{x}}$ is the **ensemble mean**.

Bergemann *et al.* (2009) showed that the **update** for the **ensemble of perturbations** can be expressed as the **solution of the** following **ODE**:

$$\frac{d\mathbf{X}}{ds} = -\frac{1}{2(M-1)} \mathbf{X}\mathbf{X}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{X} \qquad (1)$$

where the **initial condition** is the **background** $\mathbf{X}(0) = \mathbf{X}^b$, $0 \le s \le 1$ is called **pseudo-time** and the **integration yields the analysis** as $\mathbf{X}^a = \mathbf{X}(1)$. The mean is updated as in the original KF. Bergemann and Reich (2010) showed that the **full ensemble** can be **updated** as the **solution of the** following **ODE**:

$$\frac{d\overline{\overline{\mathbf{X}}}}{ds} = -\frac{1}{M-1} \mathbf{X}\mathbf{X}^T\mathbf{H}^T\mathbf{R}^{-1}\left[\frac{1}{2}\mathbf{H}\mathbf{X} + (\mathbf{H}\overline{\mathbf{x}} - \mathbf{y})\mathbf{1}^T\right] \qquad (2)$$

where $\mathbf{1} \in \Re^M$ and (2) is **integrated from** the **background** $\overline{\overline{\mathbf{X}}}(0) = \overline{\overline{\mathbf{X}}}^b$ **to** obtain the **analysis** $\overline{\overline{\mathbf{X}}}^a = \overline{\overline{\mathbf{X}}}(1)$.

Amezcua *et al.* (2012) showed that **both ODEs** can **stiffen** when the **ratio of background error covariance to observational error covariance becomes large**:

$$\beta = \left| (\mathbf{H}\mathbf{X}^b)^T \mathbf{R}^{-1} (\mathbf{H}\mathbf{X}^b) \right| / (M-1) \qquad (3)$$

This happens for **infrequent observations** and **sparsely observed areas**. A **diagonal semi-implicit (DSI)** integration **method** that **handles** this **stiffness** and is **not computationally expensive** was proposed in Amezcua *et al.* (2012).:

$$\mathbf{X}_{k+1} = \mathbf{X}_k - \frac{\Delta s}{2}\mathbf{P}_k\mathbf{H}^T\left(diag\left(\mathbf{I} + \Delta s\mathbf{H}\mathbf{P}_k\mathbf{H}^T\mathbf{R}^{-1}\right)\right)^{-1}\mathbf{R}^{-1}\mathbf{H}\mathbf{X}_k \qquad (4)$$

A similar scheme can be used for the full ensemble. Note that the **inversion is performed on a diagonal matrix**, a number of (non-uniform) steps of order $O \sim (1-10)$ yields accurate results.

The **EKBF** can be used to **assimilate quasi-continuous observations** and is amenable to **non-Gaussian extensions**. Moreover, it **can help eliminate the jumps from background to analysis**. The **forecast/assimilation process can be expressed together** in the following way:

$$\frac{d\overline{\overline{\mathbf{X}}}}{dt} = f\left(\overline{\overline{\mathbf{X}}}\right) + \sum A\left(\overline{\overline{\mathbf{X}}}\right)\delta(t - t_{assim}) \qquad (5)$$

where $A\left(\overline{\overline{\mathbf{X}}}\right)$ represents the right-hand-side of (4) and $\delta(t - t_{assim})$ is the Dirac delta centered in the assimilation times. This Dirac delta can be mollified (Bergemann and Reich 2010a) :

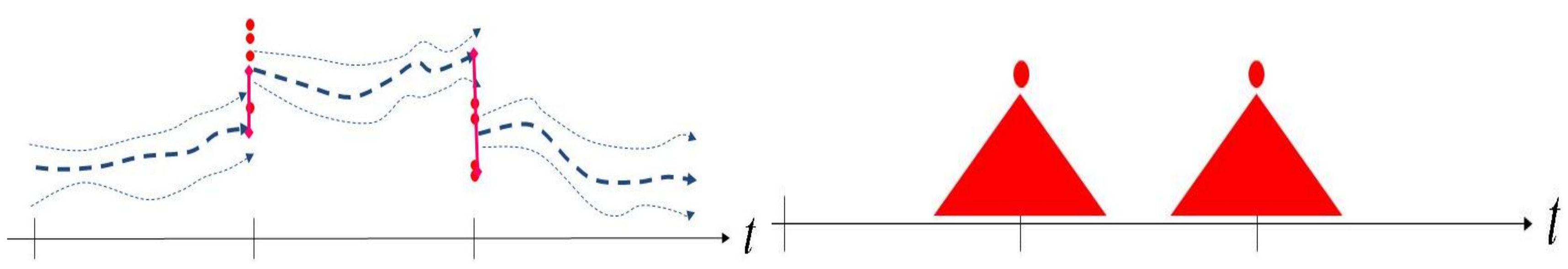

Figure 1. A typical feature of **sequential data assimilation** are the **jumps from background to analysis** (left panel). These can be **eliminated by extending the impact of observations from an instant to a finite time interval** (right panel). The **EKBF provides a suitable framework** to do this in a simple way.

Ensemble Transform Kalman-Bucy Filters

In **transform** formulations of the **EnKF**, the update is performed by a **post-multiplication** of the **ensemble with a matrix of weights** $\mathbf{W} \in \Re^{M \times M}$. For **perturbations**, the **Ensemble Transform Kalman-Bucy Filter** (ETKBF) is:

$$\frac{d\mathbf{W}}{ds} = -\frac{1}{2(M-1)} \mathbf{W}\mathbf{W}^T\mathbf{Y}^{b^T}\mathbf{R}^{-1}\mathbf{Y}^b\mathbf{W} \qquad (6)$$

with pseudo-time $0 \le s \le 1$,the initial condition $\mathbf{W}(0) = \mathbf{I}$, and the analysis $\mathbf{W}^a = \mathbf{W}(1)$ such that $\mathbf{X}^a = \mathbf{X}^b\mathbf{W}^a$. For the **full ensemble**, the Direct Ensemble Transform Kalman-Bucy filter (**DETKBF**) is:

$$\frac{d\overline{\overline{\mathbf{W}}}}{ds} = -\frac{1}{2(M-1)} \overline{\overline{\mathbf{W}}}(\mathbf{I} - \mathbf{U})\overline{\overline{\mathbf{W}}}^T\overline{\overline{\mathbf{Y}}}^{b^T}\mathbf{R}^{-1}\left[\overline{\overline{\mathbf{Y}}}^b\overline{\overline{\mathbf{W}}}(\mathbf{I} + \mathbf{U}) - 2\mathbf{y}\mathbf{1}^T\right] \qquad (7)$$

integrated from $\overline{\overline{\mathbf{W}}}(0) = \mathbf{I}$ to get $\overline{\overline{\mathbf{W}}}^a = \overline{\overline{\mathbf{W}}}(1)$ such that $\overline{\overline{\mathbf{X}}}^a = \overline{\overline{\mathbf{X}}}^b\overline{\overline{\mathbf{W}}}^a$.

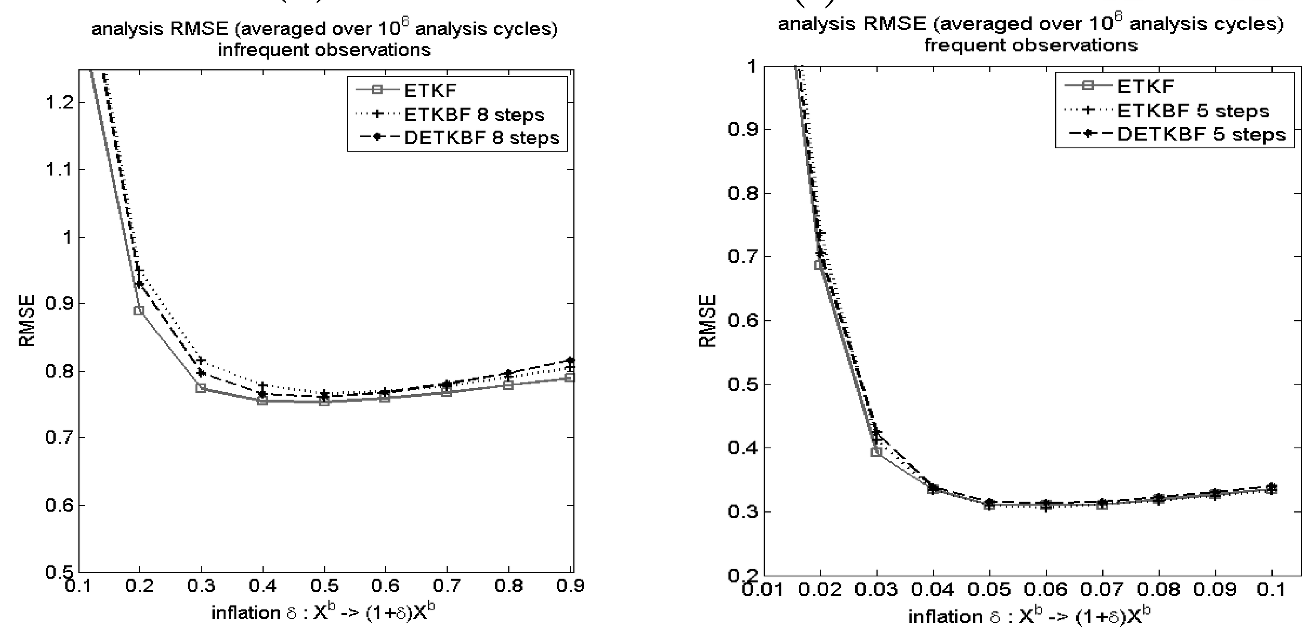

Figure 2. Results from experiments with the **3-variable Lorenz 1963 model**. These experiments use $\mathbf{H} = \mathbf{I}, \mathbf{M} = 3, \mathbf{R} = 2\mathbf{I}$., and multiplicative covariance inflation. **Two observational frequencies** are studied: every **8** integration **steps** (**frequent**) and every **25 steps** (**infrequent**), the integration step is *0.01*. The **Local Ensemble Transform Kalman Filter** (LETKF, Hunt et al., 2007) is **used as benchmark** for comparison. Both **ETKBFs achieve the performance of LETKF** with 5 pseudo-time steps (frequent obs) and 8 steps (infrequent obs).

In **transform-based formulations**, the **operations are performed in the ensemble space**, which is usually much smaller than the state space $M << N$ Moreover, **these formulations can benefit from gridpoint R-localization** (Hunt *et al.*, 2007), as well as **multiplicative adaptive covariance inflation** (Miyoshi, 2011), which **avoids manual tuning**.

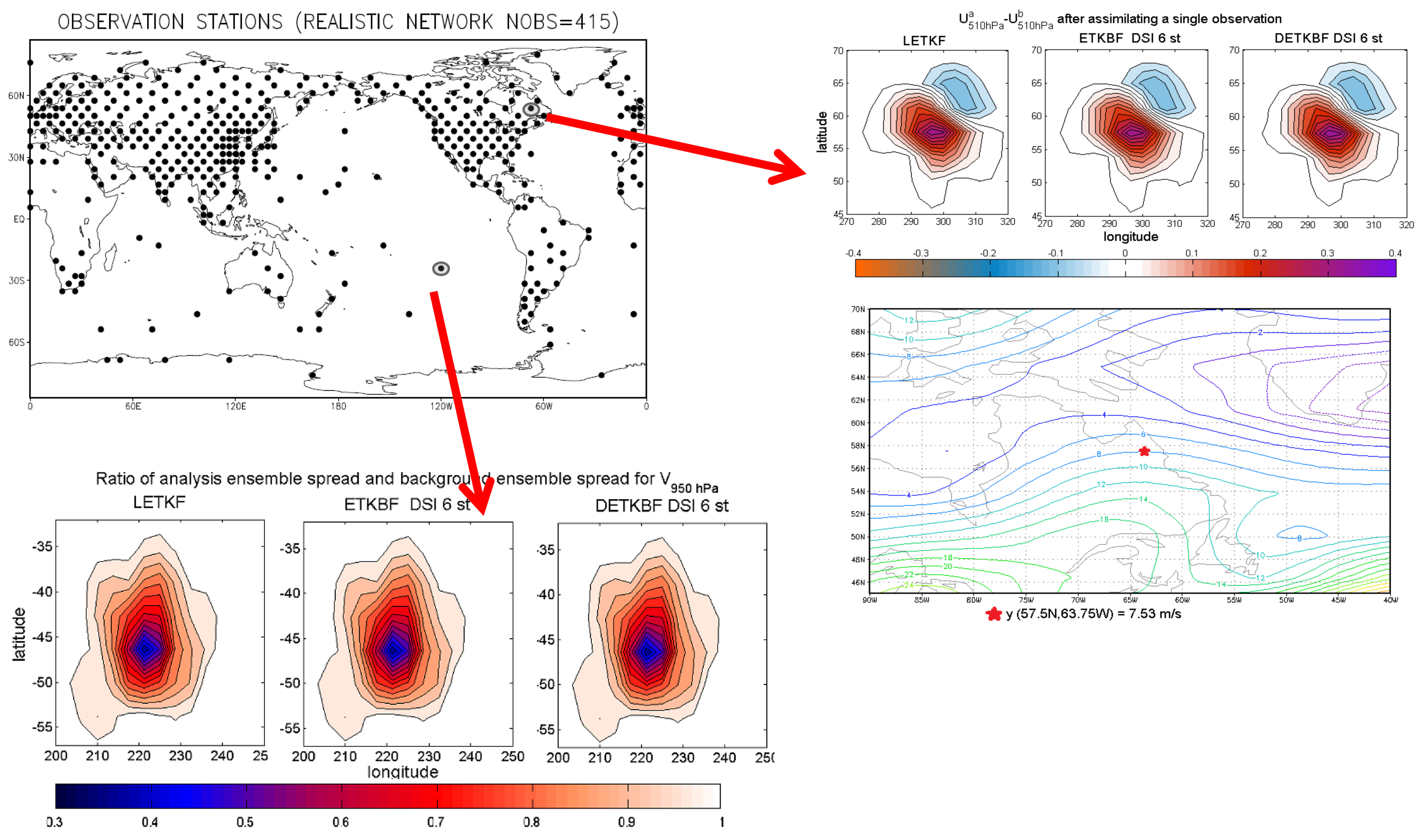

Figure 3. Use of the **ETBKFs in a medium complexity atmospheric global circulation model** (SPEEDY; Molteni, 2003). Results shown for a **well observed region** (the Labrador Peninsula) **and** a **poorly observed region** (the Southern Pacific). **In both cases, the performance of the LETKF is achieved**. Statistics computed over a longer time period for all the atmospheric variables confirm this.

Finally, the **ETKBFs can benefit from post-processing techniques developed for transform methods**. They include accurate low-resolution analyses by **weight interpolation, a no-cost smoother, forecast sensitivity to observations without adjoint model, and Running in Place/Quasi Outer-Loop** (Yang *et al.*, 2009; Kalnay and Yang 2010 ; Yang *et al.*, 2012).

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