Using a local coordinate system in a Large Eddy Simulation to account for the effects of streamline curvature

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Problem overview

- Want to simulate hurricane boundary layer (HBL)
- HBL is turbulent, so we want to resolve turbulence
  - Resolved turbulence = Large Eddy Simulation (LES)
  - LES grid spacing < 100 m
- Problem: Too expensive to run LES for an entire tropical cyclone (TC)
- Solution: Run LES for a *small portion* of TC domain

- But this is easier said than done!

LES setup

- We use NCAR's LES
- Coded in **strict Cartesian coordinates**. Why?
  - Most applications use weak, geostrophic large-scale flow
  - Lateral boundaries are periodic – necessary to solve elliptic pressure equation (via Fourier transform)*
- Large-scale flow (external pressure gradient) is prescribed via geostrophic wind
- Problem: Hurricane winds are **not** in geostrophic balance! We need to include effects of curvature (centrifugal force)

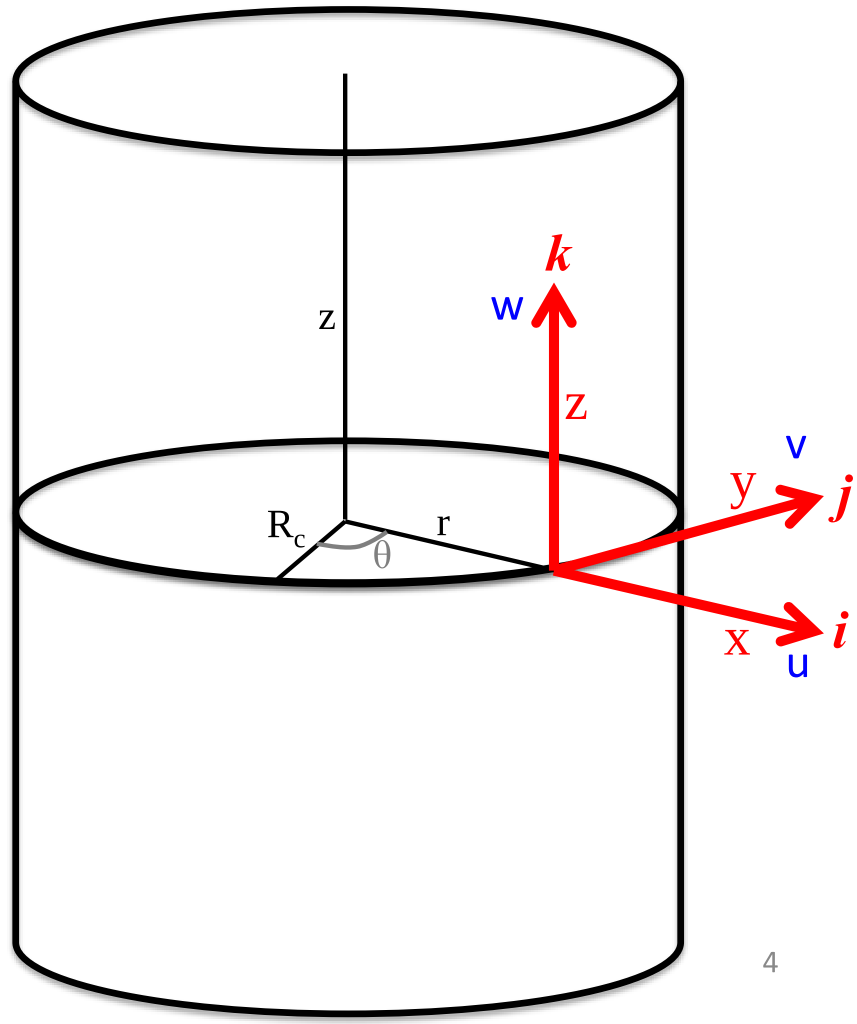
*Yicun, do you know anything about periodic boundary conditions in cylindrical coordinates?

Can NCAR LES account for curvature?

- Introduce a **local** coordinate system around a cylinder with radius R_c

- *z* direction (**k** unit vector) always points up
- *x* direction (**i** unit vector) is normal to curved surface
- *y* direction (**j** unit vector) is tangent to curved surface

- Velocity = (u, v, w)



Governing equations (1)

- Inviscid, buoyancy-free, inertial momentum eqn:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p - g\mathbf{k}$$

- In a local coordinate system, we need to account for changes in positions of unit vectors:

$$\frac{D\mathbf{v}}{Dt} = \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt}$$

- For our local coordinate system, it can be shown that

$$\frac{D\mathbf{v}}{Dt} = \mathbf{i}\left(\frac{Du}{Dt} - \frac{v^2}{r}\right) + \mathbf{j}\left(\frac{Dv}{Dt} + \frac{uv}{r}\right) + \mathbf{k}\frac{Dw}{Dt}$$

Governing equations (2)

- Momentum equations in **local** coordinates are:

$$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

- The corresponding **incompressible** continuity equation is:

$$\frac{u}{r} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Can we use this equation set?

Governing equations (3)

- In cylindrical coordinates, horizontal vorticity is conserved. We want this to be the case for local coordinates, too!

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{r}\right) = -\frac{u}{r}\frac{\partial u}{\partial y} \quad \text{where} \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{v}{r}$$

- BUT, scale analysis reveals the (u/r) terms are two orders of magnitude smaller than all other terms. Dropping these terms, the governing equations become:

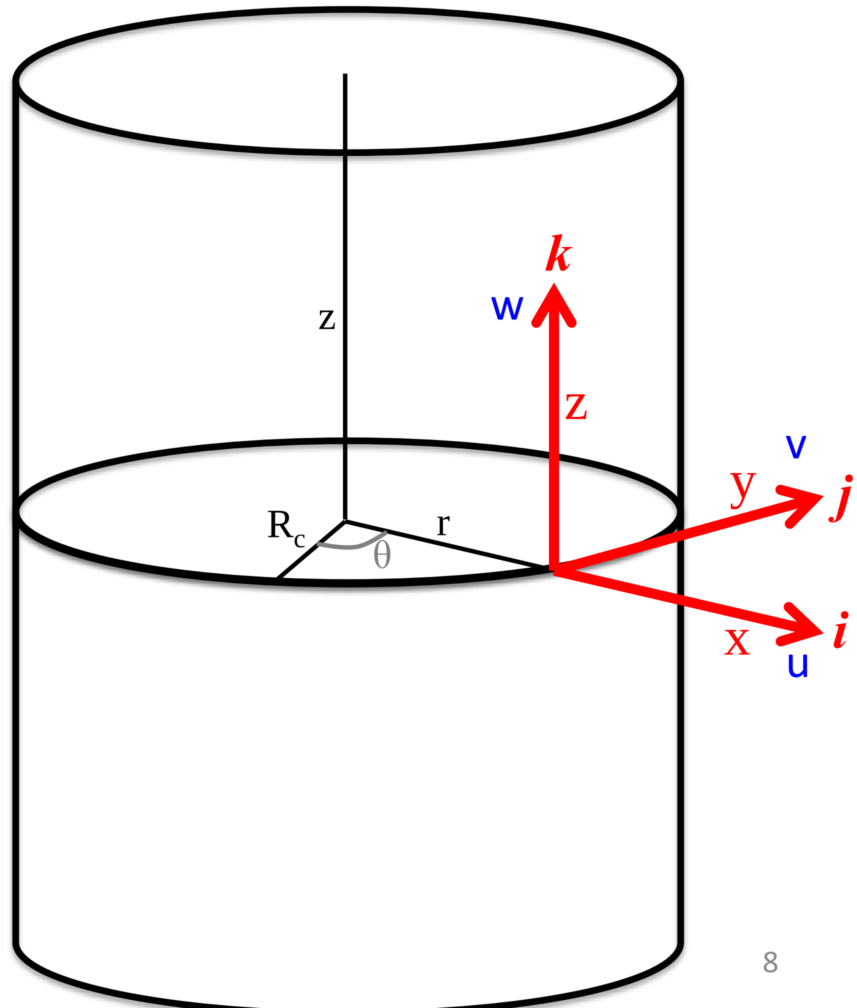
$$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

- Problem: terms with 1/r can’t be used in NCAR LES!

The 1/r terms…

- We need to use periodic boundary conditions in *x*-*y* plane
- Problem: *x* and *r* are in the same direction, so increasing *x* = increasing *r*. Thus, LES can't be periodic in *x* direction ☹
- Solution: Replace *r* by a <u>fixed</u> radius R_c. This "thin shell" approximation only works if R_c >> domain width in *r* (*x*).
  - Example: for R_c = 50 km, domain width in *x* should be ≤ 5 km
  - This approximation needs to be applied *very carefully* (I know this all too well)!



"Thin shell" approximation: $r \approx R_c$

- Momentum equations in **local** coordinates become:

$$\frac{Du}{Dt} - \frac{v^2}{R_c} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

- The corresponding **incompressible** continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- This can be coded into NCAR LES. But will there be problems?

Instability check (1)

- Linearize horizontal equations with base state (*U*, *V*, *P*) and perturbations (*u’*, *v’*, *p’*). Removing base-state-only terms and products of perturbations,

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + V\frac{\partial u'}{\partial y} - 2V\frac{v'}{R_c} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + V\frac{\partial v'}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial y}$$

- The corresponding 2-D continuity equation is:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

- Define a stream function ψ that satisfies continuity:

$$\left(u', v'\right) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right)$$

Instability check (2)

- Use stream function to find prognostic equation for vorticity $\zeta = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$:

$$\frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} + V \frac{\partial \zeta}{\partial y} = -\frac{2V}{R_c^2} \frac{\partial^2 \psi}{\partial x \partial y}$$

- Assume solutions of the form $\psi = \hat{\psi} \exp\left(ik_x x + ik_y y - i\omega t\right)$:

$$\omega - k_x U - k_y V = i \frac{2k_x k_y}{k_x^2 + k_y^2} \frac{V}{R_c^2}$$

- Instabilities occur when RHS is positive – for $V > 0$, most unstable when $k_x = k_y$ (same instability as others find)
- So this equation set is no good. The best we can do is to “filter out” the instability by replacing v/R_c with V/R_c. NCAR doesn’t like that… so I give up on this.

Summary

- We want to use LES to look at HBL
- Periodic boundary conditions mean cylindrical polar coordinate LES is not tractable
- We can adopt a **local** Cartesian coordinate system in an *attempt* to use idealized LES
- Local coordinate systems are tricky and can introduce unwanted instabilities
- The equation set used by Nakanishi and Niino (2012, JAS) is stable (not shown here), but is not popular among NCAR scientists.
- Good learning experience, but aggravating!