Relationship between Singular Vectors, Bred Vectors, 4D-Var and EnKF

Eugenia Kalnay and Shu-Chih Yang
with Alberto Carrasi, Matteo Corazza
and Takemasa Miyoshi

4th EnKF Workshop, April 2010

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Outline

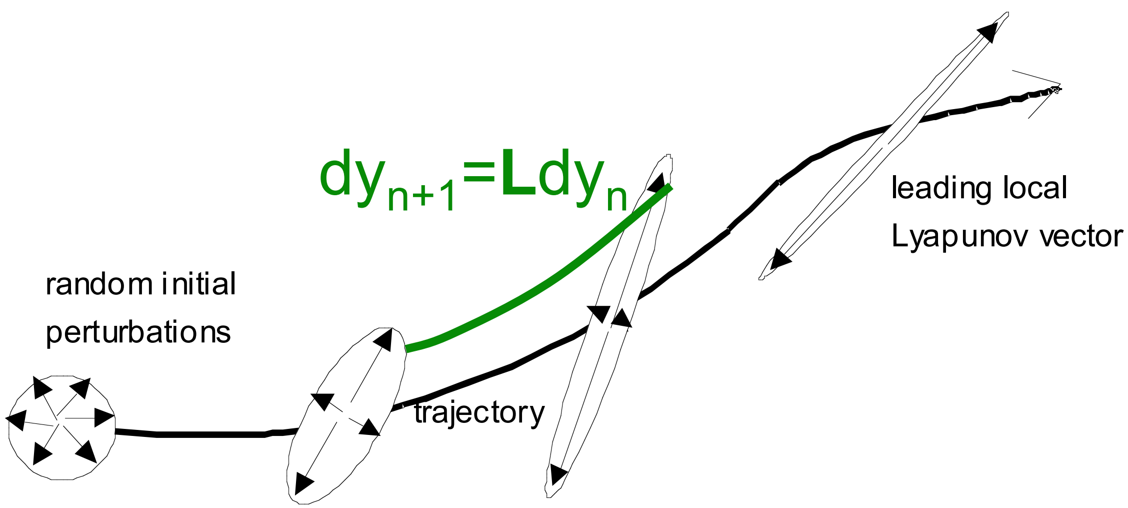
- Bred Vectors and Singular Vectors
- Dependence of initial SVs on the norm
- No-cost smoother
- Applications: Outer Loop and “Running in Place”
- Analysis increments at the end of the assimilation window: both 4D-Var and LETKF increments look like BVs.
- Analysis increments at the beginning of the assimilation window: LETKF look like BVs, 4D-Var look like SVs.

Lorenz (1965) introduced (without using their current names) all the concepts of: Tangent linear model, Adjoint model, Singular vectors, and Lyapunov vectors for a low order atmospheric model, and their consequences for ensemble forecasting.

He also introduced the concept of “errors of the day”: predictability is not constant: It depends on the stability of the evolving atmospheric flow (the basic trajectory or reference state).

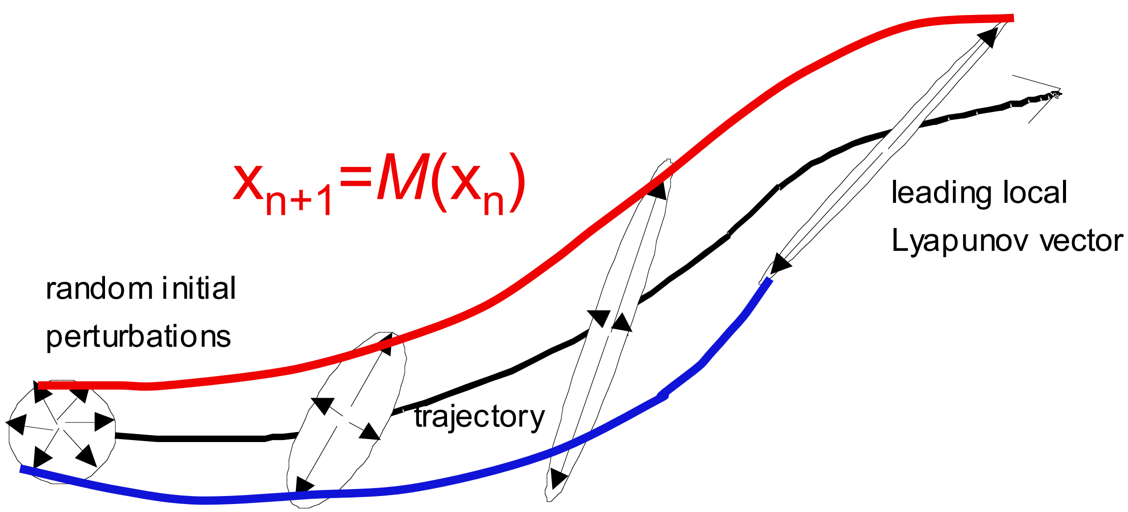
When there is an instability, all perturbations converge towards the fastest growing perturbation (leading Lyapunov Vector). The LLV is computed applying the linear tangent model **L** on each perturbation of the nonlinear trajectory

Fig. 6.7: Schematic of how all perturbations will converge towards the leading Local Lyapunov Vector



Bred Vectors: nonlinear generalizations of Lyapunov vectors, finite amplitude, finite time

Fig. 6.7: Schematic of how all perturbations will converge towards the leading Local Lyapunov Vector



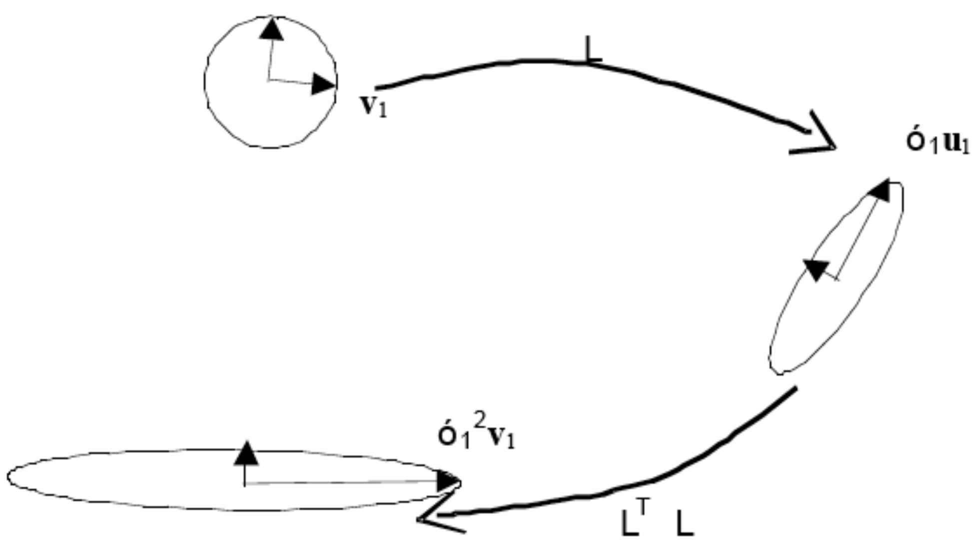
Breeding: integrate the model twice, rescale the differences periodically and add them to the control.

Apply the linear and the adjoint models

$$\mathbf{L}^T\mathbf{L}\mathbf{v}_i = \sigma_i\mathbf{L}^T\mathbf{u}_i = \sigma_i^2\mathbf{v}_i$$

So that $\mathbf{v}_i$ are the eigenvectors of $\mathbf{L}^T\mathbf{L}$ and σ_i^2 are the eigenvalues

Fig. 6.5: Schematic of the application of the TLM forward in time followed by the adjoint of the TLM to a sphere of perturbations of size 1 at the initial time.



More generally,

A perturbation is advanced from t_n to t_{n+1} $\quad \mathbf{y}_{n+1} = \mathbf{L}\mathbf{y}_n$

Find the final size with a final norm **P**:

$$\left\|\mathbf{y}_{n+1}\right\|^2 = (\mathbf{P}\mathbf{y}_{n+1})^T(\mathbf{P}\mathbf{y}_{n+1}) = \mathbf{y}_n^T\mathbf{L}^T\mathbf{P}^T\mathbf{P}\mathbf{L}\mathbf{y}_n$$

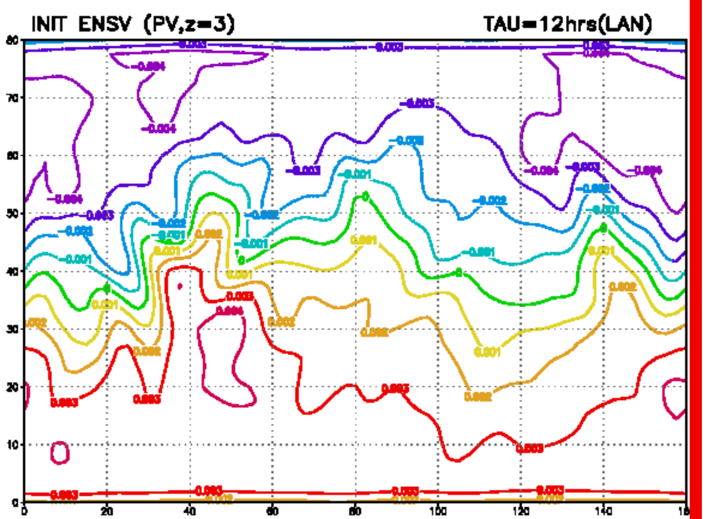
This is subject to the constraint that all the initial perturbations are of size 1 (with some norm **W** that measures the *initial* size):

$$\mathbf{y}_n^T\mathbf{W}^T\mathbf{W}\mathbf{y}_n = 1$$

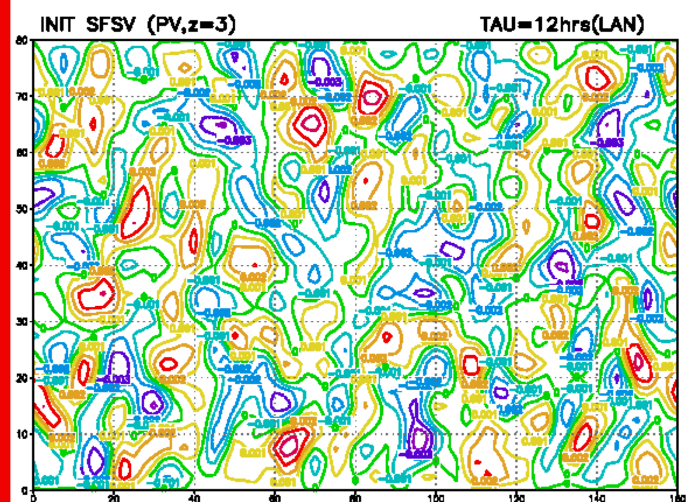
The **initial** leading SVs depend strongly on the initial norm **W** and on the optimization period $T = t_{n+1} - t_n$

QG model: Singular vectors using either enstrophy (left) or streamfunction (right) initial norms (12hr)

Initial SV with enstrophy norm

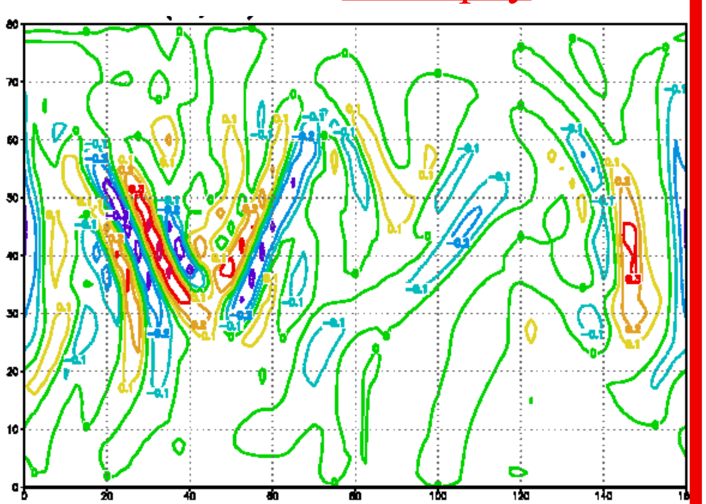


Initial SV with streamfunction norm

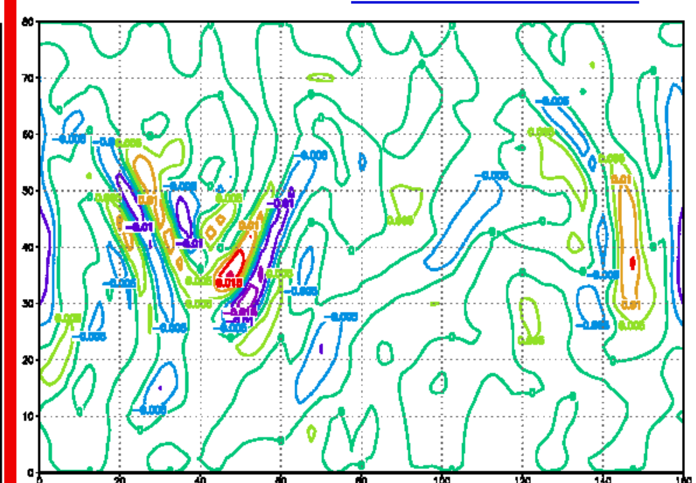


Initial SVs are very sensitive to the norm

Final SV with enstrophy norm



Final SV with streamfunction norm

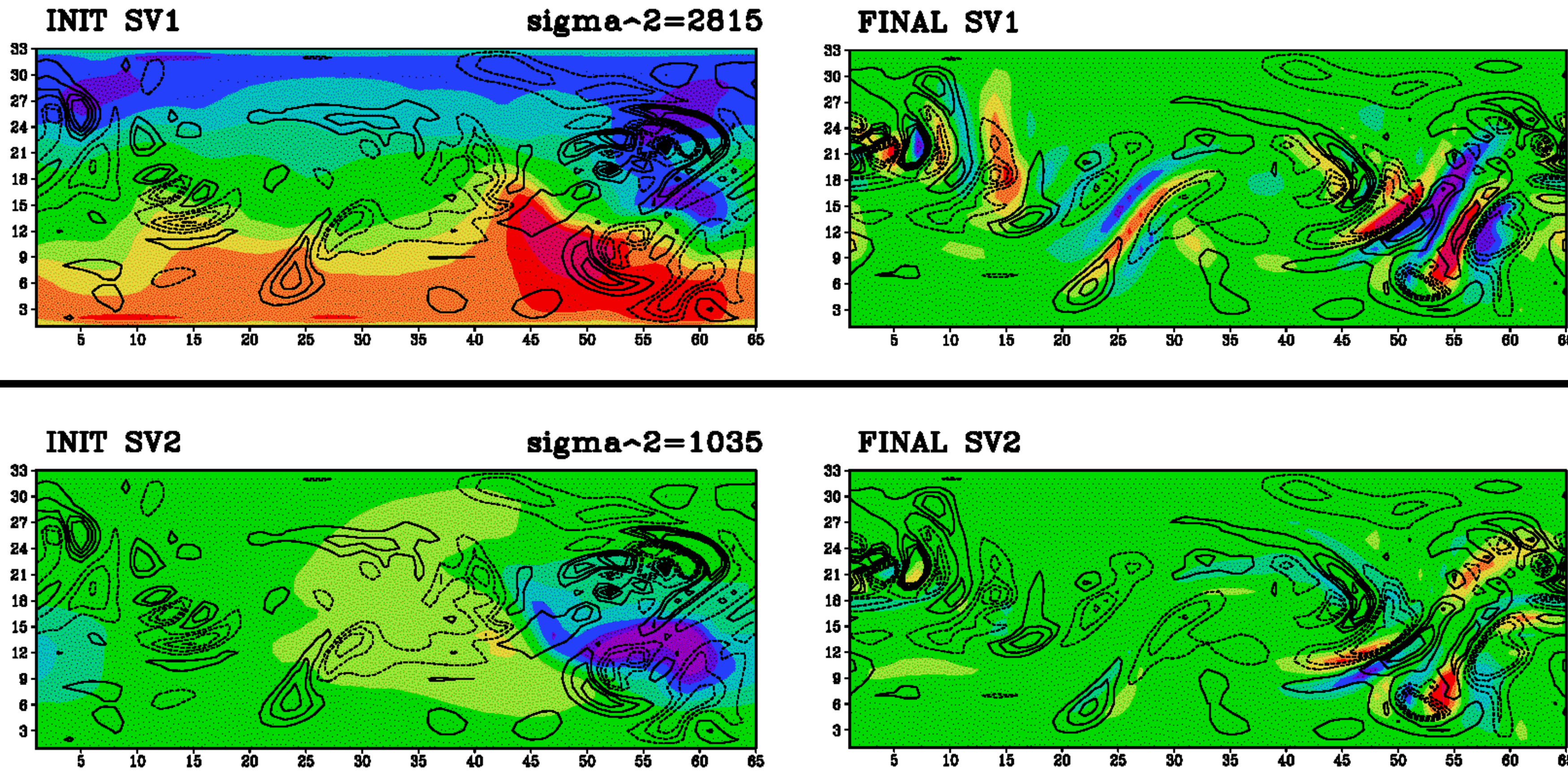


Final SVs look like bred vectors (or Lyapunov vectors)

(Shu-Chih Yang)

Two initial and final SV (24hr, vorticity^2 norm)

contours: 3D-Var forecast errors, colors: SVs

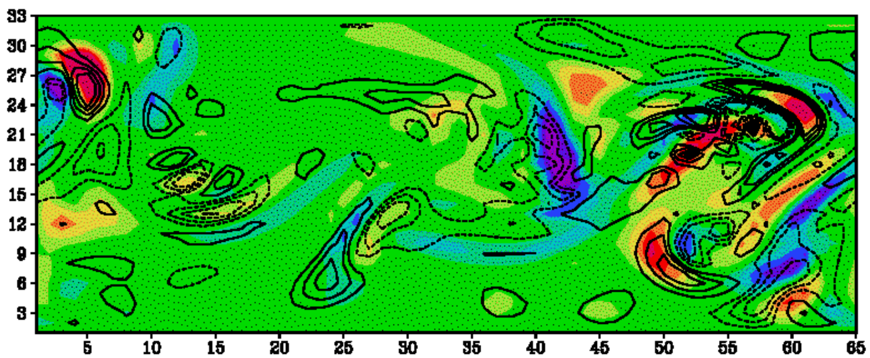


With an enstrophy norm, the initial SVs have large scales, by the end of the"optimization" interval, the final SVs look like BVs (and LVs)

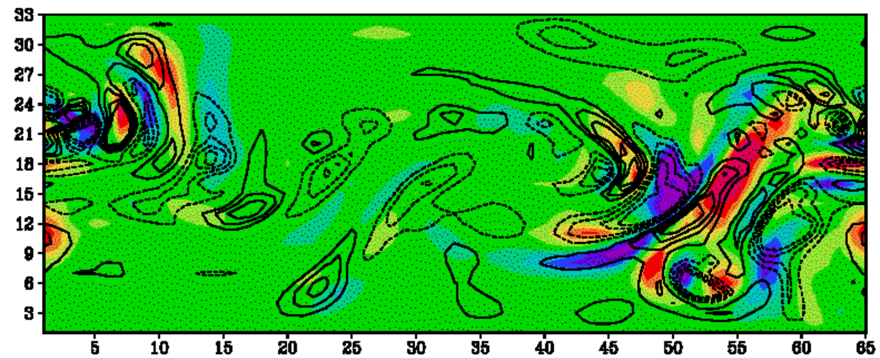
Two initial and final BV (24hr)

contours: 3D-Var forecast errors, colors: BVs

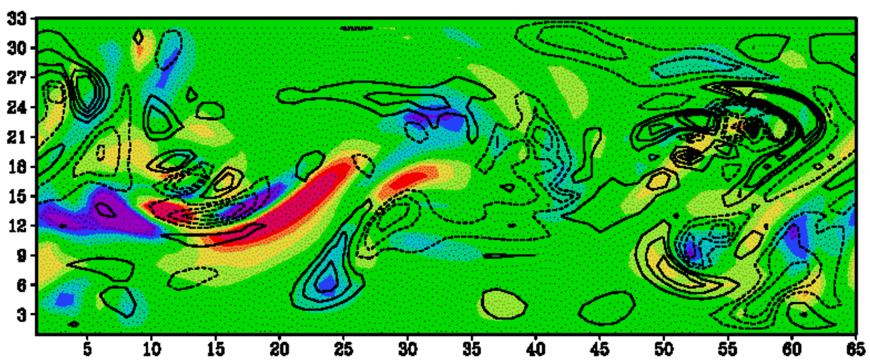
INIT BV1



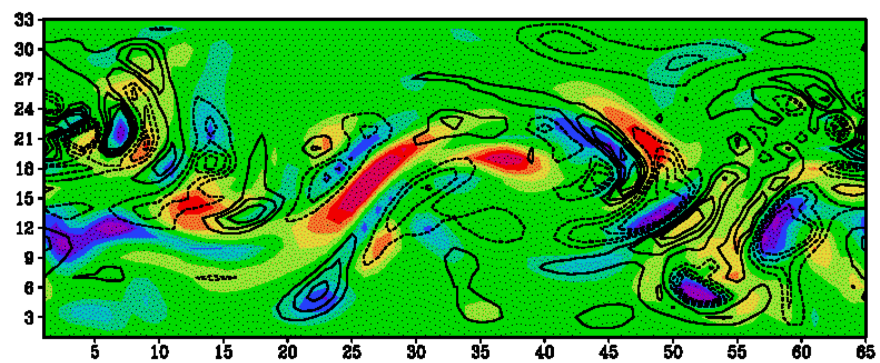
FINAL BV1



INIT BV2



FINAL BV2



The BV (colors) have shapes similar to the forecast errors (contours)

Example of nonlinear, tangent linear and adjoint codes:

Lorenz (1963) third equation: $\boxed{\dot{x}_3 = x_1 x_2 - b x_3}$

Nonlinear model, forward in time — M

$$x_3(t+\Delta t) = x_3(t) + [x_1(t)x_2(t) - b x_3(t)]\Delta t$$

Tangent linear model, forward in time — L

$$\delta x_3(t+\Delta t) = \delta x_3(t) + [x_2(t)\delta x_1(t) + x_1(t)\delta x_2(t) - b\delta x_3(t)]\Delta t$$

In the adjoint model the above line becomes — L^T

$$\delta x_3^*(t) = \delta x_3^*(t) + (1 - b\Delta t)\delta x_3^*(t+\Delta t)$$

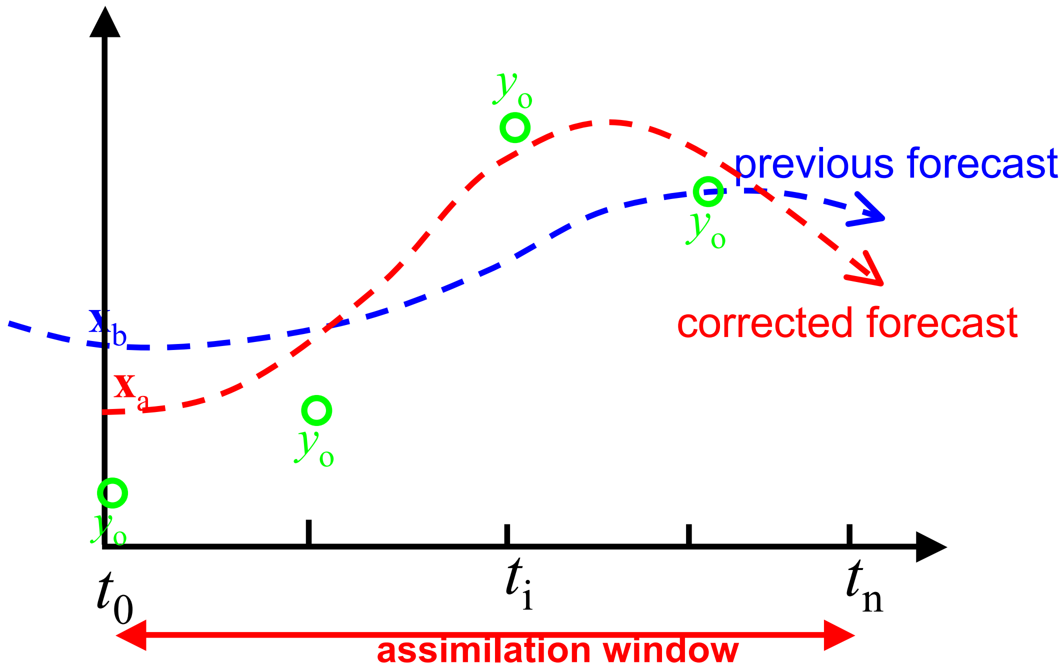
$$\delta x_2^*(t) = \delta x_2^*(t) + (x_1(t)\Delta t)\delta x_3^*(t+\Delta t)$$

$$\delta x_1^*(t) = \delta x_1^*(t) + (x_2(t)\Delta t)\delta x_3^*(t+\Delta t)$$

$$\delta x_3^*(t+\Delta t) = 0$$

backward in time

4D-Var is a smoother

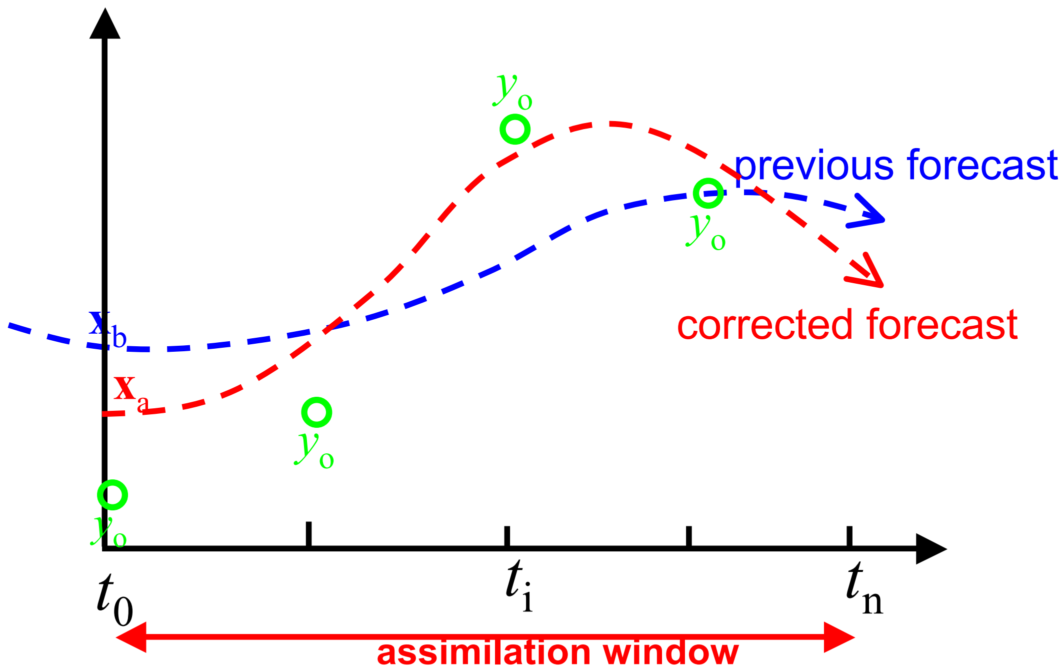


Find the initial condition such that its forecast best fits the observations within the assimilation interval

$$J(\mathrm{x}(t_0)) = \boxed{\frac{1}{2}}[\mathbf{x}(t_0)-\mathbf{x}^b(t_0)]^T \mathbf{B}_0^{-1}[\mathbf{x}(t_0)-\mathbf{x}^b(t_0)] + \boxed{\frac{1}{2}\sum_{i=0}^{i=N}}[\mathbf{y}^o_i - H(\mathbf{x}_i)]^T \mathbf{R}_i^{-1}[\mathbf{y}^o_i - H(\mathbf{x}_i)]$$

The form of the cost function suggests that the analysis increments in 4D-Var will be dominated by leading SVs.

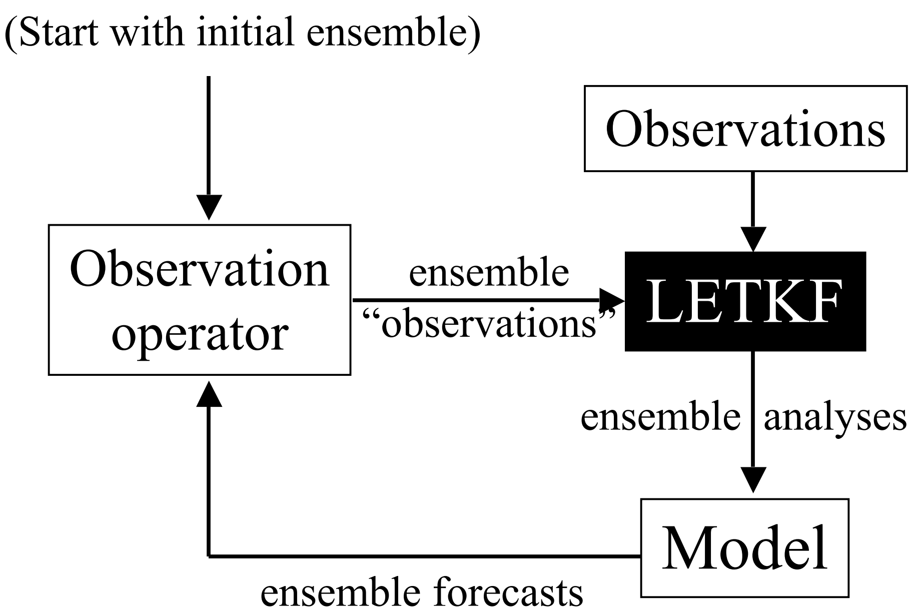
4D-Var is a smoother



The corrected forecast is the 4D-Var analysis throughout the assimilation window

What about LETKF, a sequential method?

Local Ensemble Transform Kalman Filter (Ott et al, 2004, Hunt et al, 2004, 2007) (a square root filter)



- Model independent (black box)
- Obs. assimilated simultaneously at each grid point
- Parallel analysis: each grid point is independent
- 4D LETKF extension

$$\mathbf{X}^a = \mathbf{X}^b\mathbf{T}$$

The transform matrix is a matrix of **weights.** These weights multiply the forecasts.

Local Ensemble Transform Kalman Filter (LETKF)

Globally:

Forecast step: $\mathbf{x}^b_{n,k} = M_n\left(\mathbf{x}^a_{n-1,k}\right)$

Analysis step: construct $\mathbf{X}^b = \left[\mathbf{x}^b_1 - \overline{\mathbf{x}}^b \mid \ldots \mid \mathbf{x}^b_K - \overline{\mathbf{x}}^b\right];$

$$\mathbf{y}^b_i = H(\mathbf{x}^b_i);\ \mathbf{Y}^b_n = \left[\mathbf{y}^b_1 - \overline{\mathbf{y}}^b \mid \ldots \mid \mathbf{y}^b_K - \overline{\mathbf{y}}^b\right]$$

Locally: Choose for each grid point the observations to be used, and compute the local analysis error covariance and perturbations in ensemble space:

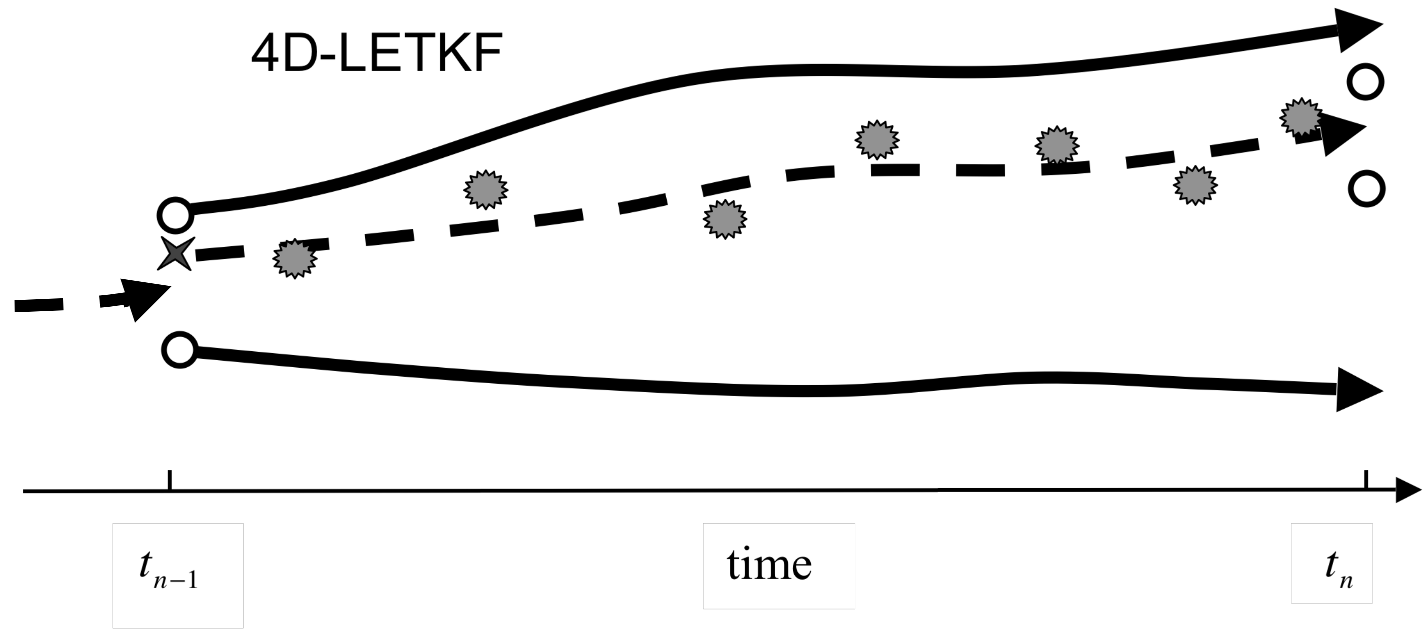
$$\tilde{\mathbf{P}}^a = \left[(K-1)\mathbf{I} + \mathbf{Y}^{bT}\mathbf{R}^{-1}\mathbf{Y}^b\right]^{-1};\ \mathbf{W}^a = [(K-1)\tilde{\mathbf{P}}^a]^{1/2}$$

Analysis mean in ensemble space: $\overline{\mathbf{w}}^a = \tilde{\mathbf{P}}^a\mathbf{Y}^{bT}\mathbf{R}^{-1}(\mathbf{y}^o - \overline{\mathbf{y}}^b)$

and add to $\mathbf{W}^a$ to get the analysis ensemble in ensemble space.

The new ensemble analyses in model space are the columns of $\mathbf{X}^a_n = \mathbf{X}^b_n\mathbf{W}^a + \overline{\mathbf{x}}^b$. Gathering the grid point analyses forms the new global analyses. Note that the the output of the LETKF are analysis weights $\overline{\mathbf{w}}^a$ and perturbation analysis matrices of weights $\mathbf{W}^a$. **These weights multiply the ensemble forecasts**.

No-cost LETKF smoother (×): apply at t_{n-1} the same weights found optimal at t_n.



Why? A linear combination of model trajectories is also a trajectory. If the trajectory is close to the truth at the end of the window, it should be close to the truth throughout the window.

Therefore the weights are valid throughout the assimilation window!

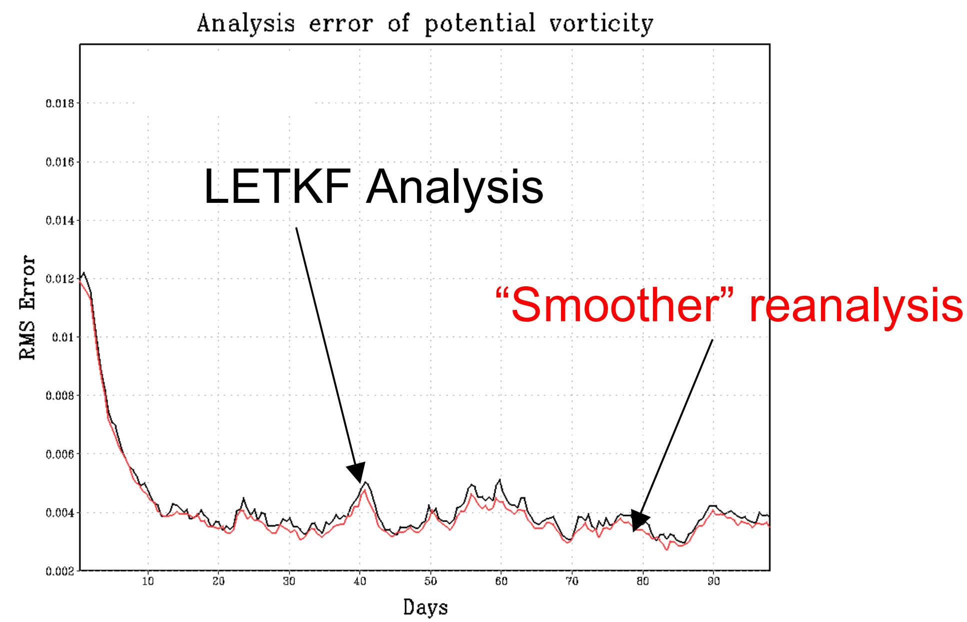
No-cost LETKF smoother tested on a QG model: it works…

LETKF analysis at time n

$$\overline{\mathbf{x}}_n^a = \overline{\mathbf{x}}_n^f + \mathbf{X}_n^f \overline{\mathbf{w}}_n^a$$

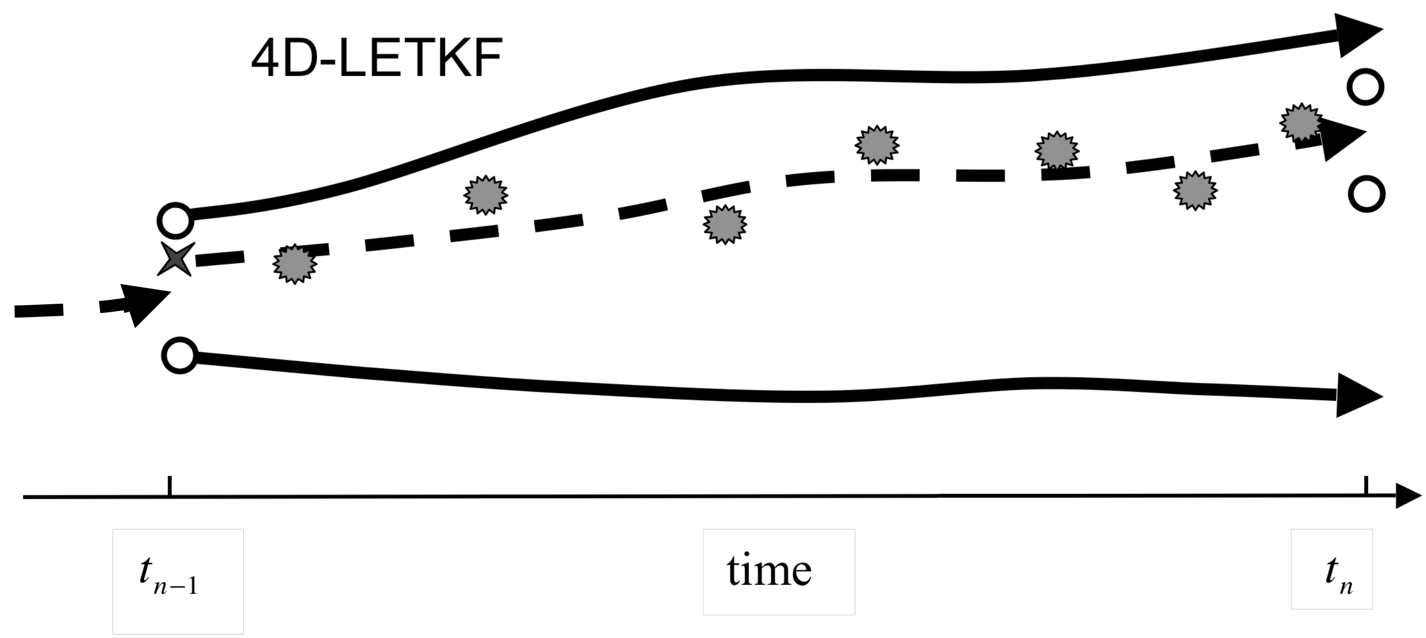
Smoother analysis at time n-1

$$\tilde{\mathbf{x}}_{n-1}^a = \overline{\mathbf{x}}_{n-1}^f + \mathbf{X}_{n-1}^f \overline{\mathbf{w}}_n^a$$



This very simple smoother allows us to go back and forth in time within an assimilation window: it allows assimilation of **future** data in reanalysis

No-cost LETKF smoother (✕): apply at t_{n-1} the same weights found optimal at t_n. It works for 3D- or 4D-LETKF



The no-cost smoother makes possible:

- ✓ Outer loop (like in 4D-Var)
- ✓ "Running in place" (faster spin-up)
- ✓ Use of future data in reanalysis
- ✓ Ability to use longer windows: dealing with nonlinearity/non-Gaussianity

Nonlinearities, “Outer Loop” and “Running in Place”

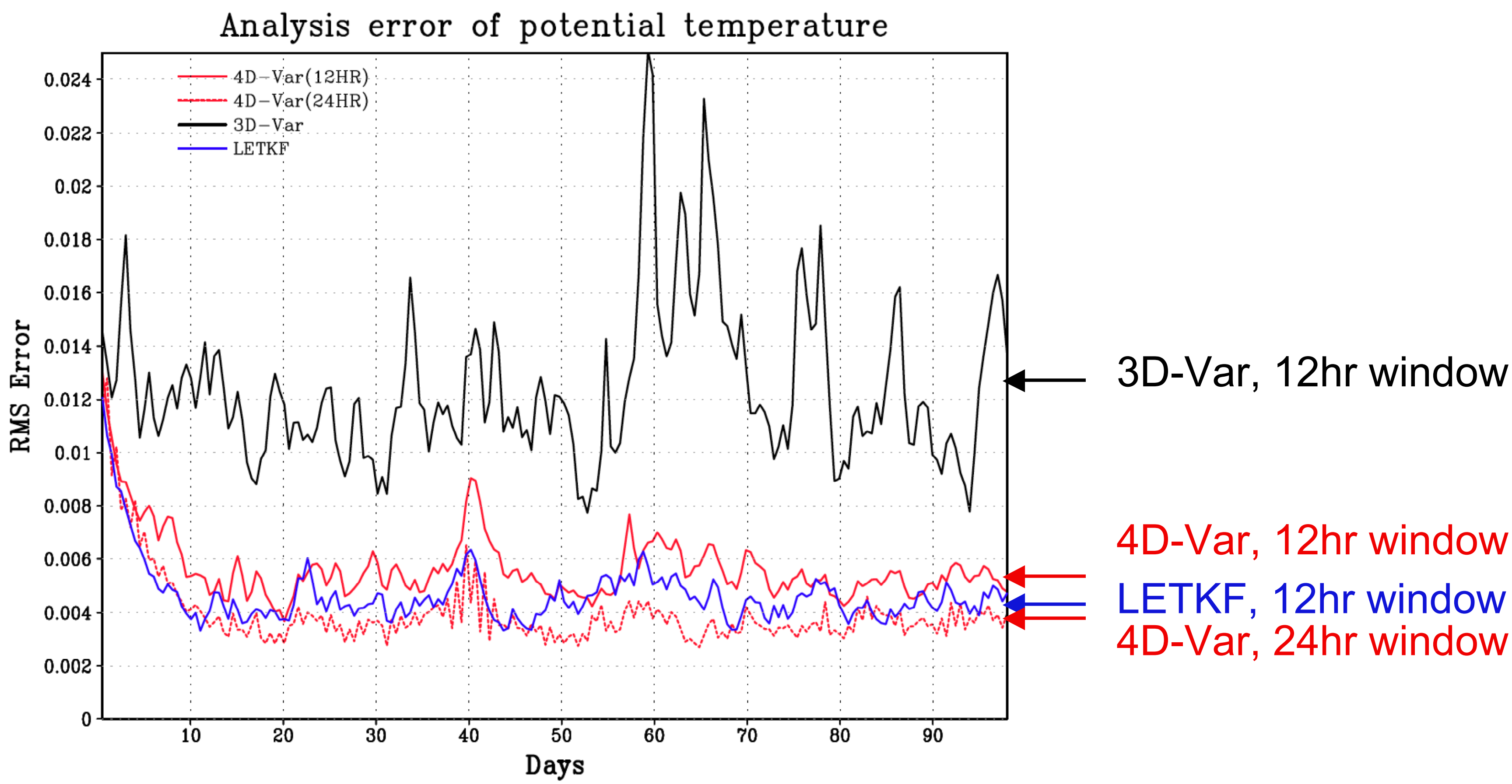
Outer loop: similar to 4D-Var: use the final weights to correct only the mean initial analysis, keeping the initial perturbations. Repeat the analysis once or twice. It re-centers the ensemble on a more accurate nonlinear solution. (Yang and Kalnay, MWR, 2010)

“Running in place”: like the outer loop but smoothing both the **analysis** and the **analysis error covariance** and iterating a few times. Accelerates the EnKF spin-up even w/o a priori information. (Kalnay and Yang, QJRMS 2010).

Lorenz -3 variable model RMS analysis error

	4D-Var	LETKF	LETKF +outer loop	LETKF +RIP
Window=8 steps	0.31	0.30	**0.27**	0.27
Window=25 steps	0.53	0.66	**0.48**	**0.39**

Comparison of 3D-Var, 4D-Var and LETKF

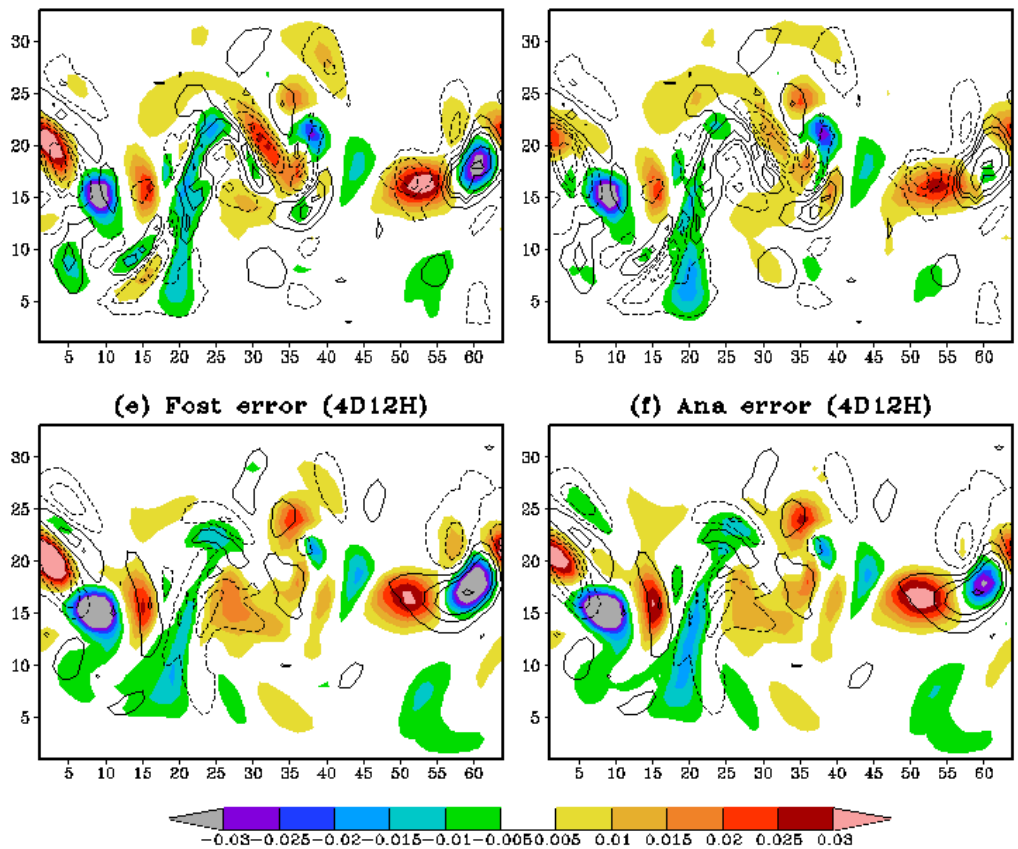


With the outer loop LETKF can also benefit from a longer window

At the end of the assimilation window, the 4D-Var and LETKF corrections are clearly very similar.

What about at the beginning of the assimilation window?

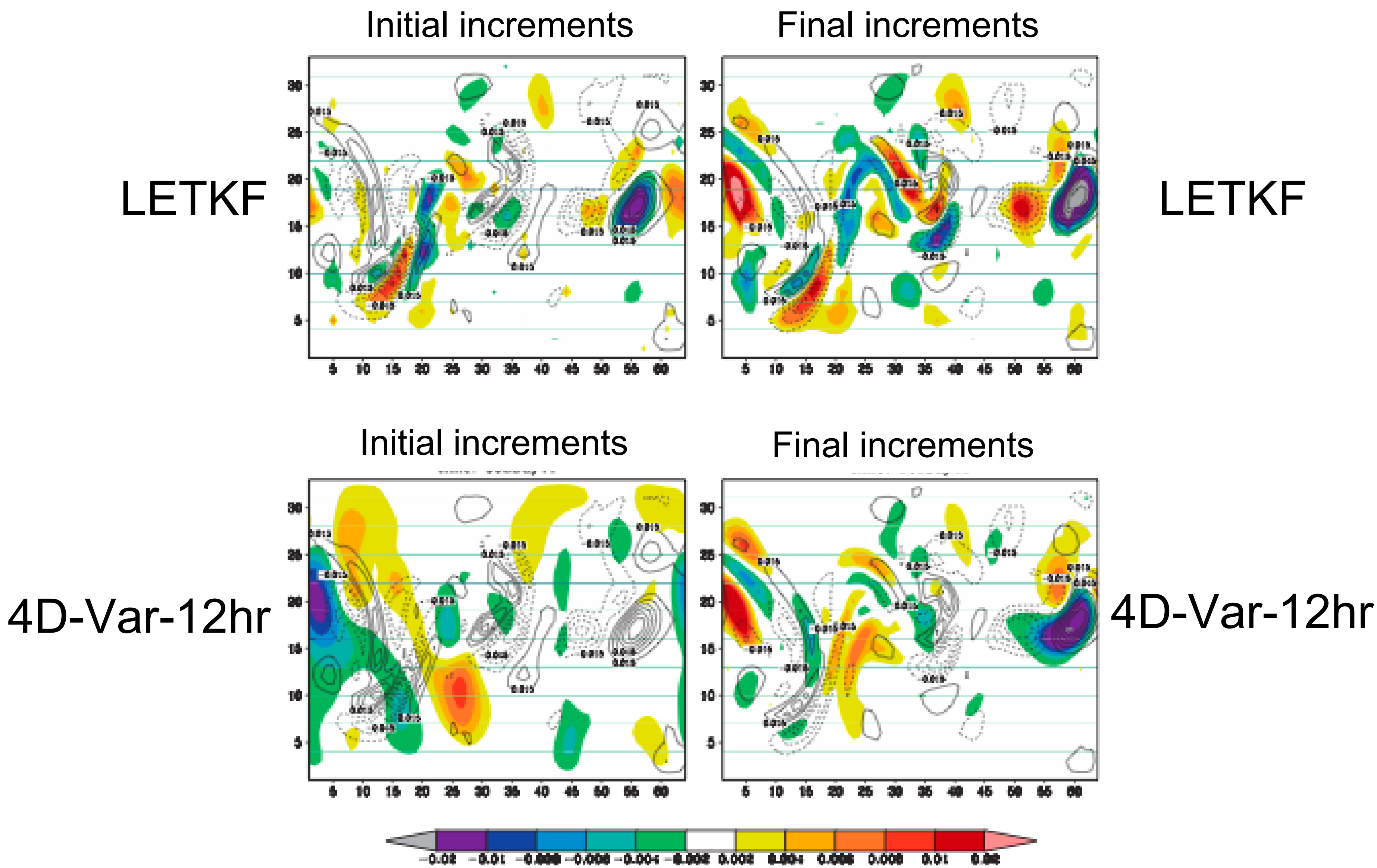
4D-Var is a smoother: we know the initial corrections.
We can use the “no-cost” LETKF smoother to get the “initial” EnKF corrections.



LETKF

4D-Var-12hr

Initial and final analysis corrections (colors), with one BV (contours)



Summary

- Initial Singular Vectors depend strongly on the norm. Bred Vectors, like leading Lyapunov vectors are norm-independent.
- Forecast errors look like BVs~LVs.
- 4D-Var is a smoother: it provides an analysis throughout the assimilation window.
- We can define a “No-cost” smoother for the LETKF.
- Applications: Outer Loop and “Running in Place”, Reanalysis.
- Analysis corrections in 4D-Var and LETKF are <u>very similar at the end of the assimilation window</u>, but <u>**very different at the beginning of the assimilation window**</u>.
- Analysis corrections at the beginning of the assimilation window look like bred vectors for the LETKF and like norm-dependent leading singular vectors for 4D-Var.

References

Kalnay et al., Tellus, 2007a (review)

Kalnay et al., Tellus, 2007b (no cost smoother)

Yang, Carrassi, Corazza, Miyoshi, Kalnay, MWR (2009) (comparison of 3D-Var, 4D-Var and EnKF, no cost smoother)

Yang and Kalnay, MWR, under revision, 2010. (Application of outer loop and RIP to handle nonlinearities and non-Gaussianities)

Kalnay and Yang, QJRMS, accepted, 2010. (Acceleration of EnKF spin-up, RIP)

Please see the “UMD Weather Chaos” web page for publications.